Lagrangians of stochastic mechanics

1990 J. Phys. A: Math. Gen. 23 3497

(http://iopscience.iop.org/0305-4470/23/15/022)

J. Phys. A: Math. Gen. 23 (1990) 3497-3510. Printed in the UK

Lagrangians of stochastic mechanics

M T Jaekel

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure†, 24, rue Lhomond, 75231 Paris Cedex 05, France

Received 7 February 1990

Abstract. The stochastic variational principle which applies to diffusions and extremises the mean value of the classical action is generalised to include non-Markovian diffusions. A whole family of different stochastic actions, labelled by a time-dependent parameter, is obtained and all of them are shown to lead to the Schrödinger equation. As a result, the well known degeneracy of the class of stochastic processes which correspond to a single quantum state is recovered. It is also related to an arbitrariness in the choice of the noise which can occur in two places: in the definition of the noise, through the time-dependent parameter, and in the auto-correlations of the noise which, although constant during the variational method, remain undetermined. The whole scheme is given a covariant form.

1. Introduction

Stochastic mechanics is a representation of quantum mechanics which, in contrast with the operator representation, preserves the ordinary commuting character of the degrees of freedom and makes the concept of trajectory play a central role (Nelson 1985). The indeterministic character of quantum mechanics is then deeply rooted in the sample paths: although continuous, these are no longer differentiable and are well described by stochastic processes. Time derivatives can still be defined, but in the mean, allowing one to obtain accelerations and to formulate a stochastic version of the Newton law. The latter has been shown to be equivalent to the Schrödinger equation, when a combination of the probability density and of the velocity field is used as the wavefunction, so that equivalence with the operator representation of quantum mechanics is ensured (Nelson 1966). However, stochasticity allows several proper accelerations to exist and the equivalence holds whenever the stochastic acceleration is correctly related to the diffusion coefficient of the process.

It was remarked early on that the stochastic representation suffers from an ambiguity: it does not assign a unique process, but infinitely many equivalent ones to the same quantum state. This degeneracy leads to problems of physical importance, if one is to interpret the stochastic processes as real physical ones and not as mere mathematical objects (Davidson 1979). In particular, the diffusion coefficient, which describes the mean 'width' of the trajectories, is not determined but can be scaled by an arbitrary parameter, provided the stochastic acceleration is chosen accordingly. Surely, the early noticed connection between diffusions and Riemannian manifolds

† Laboratoire propre du Centre National de la Recherche Scientifique, associé à l'Ecole Normale Supérieure et à l'Université de Paris-Sud.

(Itô 1975) allows one to exploit symmetry requirements, like covariance under (time-dependent) changes of coordinates, and to relate the diffusion coefficients to the metrics of the underlying Riemannian manifold (Dohrn and Guerra 1978, Nelson 1985). But it still fails to fix the scaling parameter, which can even have an arbitrary time dependence (Jaekel 1988). On another hand, although for technical convenience the process is often considered to be Markovian, there is at present no physical justification for such a hypothesis.

Nonetheless, one can still hope to raise at least the scaling ambiguity, if one manages to justify a unique choice for the stochastic acceleration, or equivalently, for the stochastic action from which the Schrödinger equation can be derived. Indeed, among the various stochastic variational principles developed up to now (Yasue 1981, Guerra and Morato 1983, Morato 1985, Misawa and Yasue 1987), the scheme introduced by Guerra and Morato seems to fulfil such a hope. In the latter, one obtains the stochastic action by computing the mean value of the classical action over the trajectories of a Markovian diffusion. This direct identification with the classical action thus leads to a unique stochastic action, which, when extremised, can be seen to provide the stochastic acceleration with the scaling parameter set equal to 1.

This paper has two main aims. Firstly, to give a generalisation of Guerra and Morato's scheme for non-Markovian diffusions, which preserves its main feature, i.e. its direct (computable) connection with the classical action. This will result, in particular, in the restoration of the scaling ambiguity, ruining the hope just expressed, but will also exhibit its relation to the Markov hypothesis. Secondly, to put into evidence a deep connection between the properties of the noise, which characterise the diffusion process, and the constraints which define the variational principle. This will also provide a new insight into the nature of the degeneracy of the processes corresponding to a single quantum state. The extension to non-Markovian diffusion will produce a further result: the scaling ambiguity will be related to the definition of the noise, and traced back to a choice in its time prescription which, although different in nature, is very similar to the convention which is known to exist in stochastic calculus (in particular, between Itô and Stratonovich calculus (Schuss 1980)). A covariant formulation of these properties, respecting the underlying geometry, will also be given. In particular, the action will be identified with an invariant length for the trajectories, obtained with the use of a generalised stochastic parallel transport.

For the sake of clarity, there will be two main parts, which will follow the two conceptual steps of Guerra and Morato's scheme. A first one will be devoted to an extension of the derivation of the Schrödinger equation from a variational principle, that will apply to non-Markovian diffusion and to a family of stochastic actions. The latter will then be obtained in the next part, by computing the mean value of the classical action over the trajectories of the diffusions, or more precisely, of a proper extension of the classical action which will take into account the corrections due to the necessary Itô terms (MacLaughlin and Schulman 1971).

2. The stochastic variational principle

The direct connection with the classical action provides one reason for dealing preferably with diffusion processes, and thus for using the variational method developed by Guerra and Morato (1983), (which has also been extended to pathwise variations (Morato 1985, Marra 1987). A different characterisation of the stochastic processes, as with semi-martingales, would call for another variational approach, such as that developed by Yasue (1981), Misawa and Yasue (1987). Another reason is the close relationship between diffusions and Riemannian manifolds (Itô 1975, Nelson 1985), which naturally endows them with intrinsic geometrical properties, allowing one to develop a manifestly covariant formulation. In this context, the action will also be identified with an invariant length for the trajectories.

We shall begin by recalling a few definitions, which are extensively used in the stochastic representation.

2.1. Diffusion processes

Diffusions are defined as collections of time-indexed random variables: x^i_t, $i = 1 \ldots N$, with the following properties:

$$\begin{aligned} &\mathrm{d}x^i_t = x^i_{t+\mathrm{d}t} - x^i_t \\ &\langle \mathrm{d}x^i_t \rangle_t = b^i(x_t)\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t) \\ &\langle \mathrm{d}x^i_t\,\mathrm{d}x^j_t \rangle_t = 2\nu^{ij}(x_t)\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t) \\ &\langle \mathrm{d}x^{i_1}_t\,\mathrm{d}x^{i_2}_t \ldots \mathrm{d}x^{i_n}_t \rangle_t = \mathrm{o}(\mathrm{d}t) \qquad n > 2. \end{aligned} \tag{1}$$

$\langle\ \rangle_t$ will denote, in the following, the expectation value of any random variables defined on the same probability space, taken conditionally in the values x^i_t, at time t, for the variables x^i. The probability density will satisfy in particular the Fokker–Planck equation:

$$\partial_t \rho + \nabla_i(b^i \rho) - \nabla_i \nabla_j(\nu^{ij}\rho) = 0 \qquad \nabla_i = \partial/\partial x^i. \tag{2}$$

Let us note that the drift and diffusion fields b and ν, as defined in (1), together with the probability density ρ, are not sufficient to specify the diffusion process: only if one further restricts the diffusion process to be Markovian, is the latter uniquely determined. Quite generally, one can define a noise w through the following stochastic differential equation, which is the rigorous form of the Langevin equation, and the basis of Itô calculus (Schuss 1980):

$$\mathrm{d}x^i_t = b^i(x_t)\,\mathrm{d}t + \mathrm{d}w^i_t + \mathrm{o}(\mathrm{d}t) \tag{3}$$

or else:

$$x^i_t - x^i_{t_0} = w^i_t - w^i_{t_0} + \int_{t_0}^{t} b^i(x_s)\,\mathrm{d}s.$$

Thus, the diffusion is specified by the drift field b and the noise w. The latter is only subject to the restrictions:

$$\langle \mathrm{d}w^i_t \rangle_t = \mathrm{o}(\mathrm{d}t) \qquad \langle \mathrm{d}w^i_t\,\mathrm{d}w^j_t \rangle_t = 2\nu^{ij}(x_t)\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t)$$

and can still span a large variety of different diffusions. Restricting the noise to martingales, i.e.

$$\langle \mathrm{d}w^i_t \rangle_{s \leqslant t} = 0$$

is equivalent to restricting to Markovian diffusion processes, but as will be seen, is by no means necessary.

The diffusive nature (1) allows one to define time derivatives in the mean:

$$\mathrm{d}f_t = f_{t+\mathrm{d}t} - f_t \qquad \mathrm{D}f_t = \lim_{\mathrm{d}t\to 0^+} \frac{\langle \mathrm{d}f_t\rangle_t}{\mathrm{d}t}$$

$$\mathrm{d}_* f_t = f_t - f_{t-\mathrm{d}t} \qquad \mathrm{D}^* f_t = \lim_{\mathrm{d}t\to 0^+} \frac{\langle \mathrm{d}_* f_t\rangle_t}{\mathrm{d}t}$$

so that in particular, for any function of the variables x_t^i:

$$\mathrm{D} = \partial_t + b^i \nabla_i + \nu^{ij}\nabla_i\nabla_j$$

$$\mathrm{D}^* = \partial_t + b^{*i}\nabla_i - \nu^{ij}\nabla_i\nabla_j$$

with

$$\mathrm{D}x^i = b^i \qquad \mathrm{D}^* x^i = b^{*i} \qquad b^{*i} = b^i - 2\nabla_j(\nu^{ij}\rho)/\rho .$$

In fact, these time derivatives generate a whole affine set, which reflects the covariance of the diffusion process (1) under time-dependent changes of variables:

$$\mathrm{D}_\lambda = \frac{1+\lambda}{2}\mathrm{D} + \frac{1-\lambda}{2}\mathrm{D}^*$$

when applied to a function of x_t^i:

$$\mathrm{D}_\lambda = \partial_t + b_\lambda^i \nabla_i + \lambda \nu^{ij}\nabla_i\nabla_j$$

with

$$b_\lambda^i = \mathrm{D}_\lambda x^i = \frac{1+\lambda}{2} b^i + \frac{1-\lambda}{2} b^{*i}$$

so that

$$\langle f_{t'}\rangle - \langle f_t\rangle = \left\langle \int_t^{t'} \mathrm{D}_\lambda f_s \, \mathrm{d}s \right\rangle . \tag{4}$$

Indeed, these time derivatives correspond to different ways of taking the conditional expectation in the mean time derivative:

$$\mathrm{D}_\lambda f_t = \lim_{\mathrm{d}t\to 0^+} \frac{\langle \mathrm{d}_\lambda f_t\rangle_t}{\mathrm{d}t}$$

$$\mathrm{d}_\lambda f_t = \mathrm{d}f_{t_\lambda} \qquad t_\lambda = t - \frac{1-\lambda}{2}\mathrm{d}t \qquad (\lambda(t)).$$

The arbitrary time-dependent parameter λ thus reflects a general infinitesimal arbitrariness in time specification. This then allows the definition of different λ-noises w_λ according to

$$\begin{aligned} x_t^i - x_{t_0}^i &= w_{\lambda t}^i - w_{\lambda t_0}^i + \int_{t_0}^{t} b_\lambda^i(x_s)\,\mathrm{d}s \\ \mathrm{d}x_t^i &= \mathrm{d}w_{\lambda t}^i + \int_t^{t+\mathrm{d}t} b_\lambda^i(x_s)\,\mathrm{d}s. \end{aligned} \tag{5}$$

For any choice of λ, the diffusion process is determined by the corresponding drift b_λ and noise w_λ, where the latter is only constrained by

$$\langle \mathrm{d}_\lambda w_{\lambda t}^i\rangle_t = \mathrm{o}(\mathrm{d}t) \qquad \langle \mathrm{d}w_{\lambda t}^i \, \mathrm{d}w_{\lambda t}^j\rangle_t = 2\nu_t^{ij}\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t).$$

One will have noted that λ varies between 1 and -1, and that different differentials d_λ correspond to different conventions in stochastic calculus, where $\lambda=1$ is the Itô convention and $\lambda=0$ the Stratonovich one (Schuss 1980). However, it must be emphasised that the use made in the conditional expectation values, and consequently in the definition of the noise (5), is a new feature, to be distinguished from the use in the definition of stochastic integrals. The difference will appear more clearly in section 3.

2.2. *Variational principle*

A very clear and detailed description of Guerra and Morato's scheme has been given by Nelson (1985), so that its generalisation will be given here in a concise form, with comments on the new features only.

The diffusion process is assumed to describe a dynamical system whose classical Lagrangian is known:

$$\mathscr{L}=\tfrac{1}{2}m_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}+a_i\frac{\mathrm{d}x^i}{\mathrm{d}t}-\mathscr{V}. \tag{6}$$

Accordingly, the following actions are defined for any diffusion process:

$$\mathscr{S}_\lambda=\left\langle\int_t^{t'}\mathscr{L}_\lambda\,\mathrm{d}s\right\rangle \tag{7}$$

with the stochastic Lagrangians:

$$\mathscr{L}_\lambda=\tfrac{1}{2}m_{ij}\tilde{b}^i_\lambda\tilde{b}^j_\lambda+a_i\tilde{b}^i_\lambda+\frac{\hbar}{2}\mathrm{D}_i(\tilde{b}^i_\lambda+a^i)-\mathscr{V}\qquad a^i=m^{ij}a_j. \tag{8}$$

These stochastic actions will be obtained from (6) in section 3.

λ is an arbitrary time-dependent parameter (Guerra and Morato take $\lambda=1$), which will label the whole variational method. For covariance requirements, the following quantities have been introduced:

$$\tilde{b}^i_\lambda=b^i_\lambda+\lambda\nu^{jk}\Gamma^i_{jk}\qquad \Gamma^i_{jk}=\frac{\nu^{il}}{2}(\nabla_j\nu_{kl}+\nabla_k\nu_{jl}-\nabla_l\nu_{jk})\qquad \mathrm{D}_iA^j=\nabla_iA^j+\Gamma^i_{jk}A^k$$

(Γ^i_{jk} is the Christoffel symbol associated with the metrics ν, and D_i the corresponding covariant derivative). From (7) and (8) it is clear that the stochastic action is totally determined by the probability density ρ, the drift field $\tilde{b}_\lambda$ and the diffusion field ν. The latter will furthermore be chosen as

$$\nu^{ij}=\frac{\hbar}{2\lambda}m^{ij}. \tag{9}$$

This will appear to be the most general choice compatible with the Schrödinger equation. Let us just remark for the moment, that for any choice of λ, the stochastic action is easily seen to transform like a scalar function, under time-dependent changes of variables (as $\tilde{b}_\lambda$ and $-a$ transform like velocities, and $\mathscr{V}+\frac{1}{2}a_ia^i$ like scalar (Jaekel 1988)).

The variational principle can now be stated in the following way: ν does not vary, but is fixed once and for all (and so also is λ); $\tilde{b}_\lambda$ and ρ are varied, but with their values at the endpoints kept fixed:

$$\delta\tilde{b}_{\lambda t}=\delta\tilde{b}_{\lambda t'}=0\qquad \delta\rho_t=\delta\rho_{t'}=0\qquad \delta\nu^{ij}=0. \tag{10}$$

The critical diffusions will be those for which the stochastic actions are stationary, i.e. do not vary at first order in $\delta\rho$ and $\delta\tilde{b}_\lambda$:

$$\delta\mathcal{S}_\lambda = \mathrm{o}(\delta\tilde{b}_\lambda, \delta\rho).$$

Writing these conditions will give differential equations characterising the critical diffusions.

For that purpose, let us introduce the following function $S(t, x)$, defined as a solution of

$$\mathrm{D}_\lambda S = \mathcal{L}_\lambda$$

that is:

$$\partial_t S + \tilde{b}^i_\lambda \mathrm{D}_i S + \lambda \nu^{ij} \mathrm{D}_i \mathrm{D}_j S = \frac{\hbar}{4\lambda} \nu_{ij} \tilde{b}^i_\lambda \tilde{b}^j_\lambda + a_i \tilde{b}^i_\lambda + \frac{\hbar}{2} \mathrm{D}_i(\tilde{b}^i_\lambda + a^i) - \mathcal{V}. \tag{11}$$

It might be worth noticing that the only difference with Guerra and Morato's scheme, which will allow for non-Markovian diffusions, appears here: in their case, S is defined as:

$$S = -\left\langle \int_t^{t'} \mathcal{L}_\lambda \, \mathrm{d}s \right\rangle_t \tag{11$'$}$$

while here only (11) is used. For Markovian diffusions, and $\lambda = 1$ only, (11) can be seen to result from (11′).

It is then easy to see that according to (4), S provides the action through

$$\mathcal{S}_\lambda = \langle S_{t'} \rangle - \langle S_t \rangle$$

so that the variation can be rewritten

$$\delta\mathcal{S}_\lambda = \langle \delta S_{t'} \rangle - \langle \delta S_t \rangle$$

(where because of (10), $\langle\ \rangle$ is the expectation value on the unvaried process). One then easily derives successively, from (4):

$$\begin{aligned}
\delta\mathcal{S}_\lambda &= \left\langle \int_t^{t'} \mathrm{D}_\lambda \delta S_s \, \mathrm{d}s \right\rangle \\
&= \left\langle \int_t^{t'} [\delta\mathcal{L}_\lambda - (\delta \mathrm{D}_\lambda) S_s] \, \mathrm{d}s \right\rangle + \mathrm{o}(\delta\tilde{b}_\lambda) \qquad \text{(from (11))} \\
&= \left\langle \int_t^{t'} \left[m_{ij}\tilde{b}^j_\lambda + a_i - \frac{\hbar}{2}\nabla_i \ln\tilde{\rho} - \nabla_i S \right] \delta\tilde{b}^i_\lambda \, \mathrm{d}s \right\rangle + \mathrm{o}(\delta\tilde{b}_\lambda) \qquad \text{(from (8) and (4))}
\end{aligned}$$

(where $\tilde{\rho} = \rho|\nu|^{1/2}$ and $|\nu| = |\det \nu^{ij}|$) giving the following characterisation of critical processes:

$$\nabla_i S = m_{ij} v^j + a_i \qquad v^i = \frac{b^i + b^{*i}}{2} = \frac{\tilde{b}^i_\lambda + \tilde{b}^i_{-\lambda}}{2}. \tag{12}$$

Then (11) and (12) can be rewritten

$$\partial_t S + \tfrac{1}{2} m^{ij}(\nabla_i S - a_i)(\nabla_j S - a_j) - \frac{\hbar^2}{2} \frac{m^{ij}\mathrm{D}_i \mathrm{D}_j \tilde{\rho}}{\tilde{\rho}} + \mathcal{V} = 0$$

the gradient of which is

$$\tfrac{1}{2}[\mathrm{D}_\lambda(m_{ij}\tilde{b}^j_{-\lambda}) + \mathrm{D}_{-\lambda}(m_{ij}\tilde{b}^j_\lambda)] + (\nabla_j a_i - \nabla_i a_j)\frac{\tilde{b}^j_\lambda + \tilde{b}^j_{-\lambda}}{2} + \nabla_i \mathcal{V} - \partial_t a_i = 0$$

also giving the characterisation of the critical diffusions in the form of a stochastic Newton law. Indeed, one can recall the classical Newton law resulting from the classical Lagrangian (6):

$$\frac{\mathrm{D}}{\mathrm{d}t}\left(\frac{m_{ij}\,\mathrm{d}x^j}{\mathrm{d}t}\right)+(\nabla_j a_i-\nabla_i a_j)\frac{\mathrm{d}x^j}{\mathrm{d}t}+\nabla_i\mathcal{V}-\partial_t a_i=0 \qquad \frac{\mathrm{D}A_i}{\mathrm{d}t}=\frac{\mathrm{d}A_i}{\mathrm{d}t}-\Gamma^k_{ij}A_k\frac{\mathrm{d}x^j}{\mathrm{d}t}.$$

On another hand, when coupled to the Fokker–Planck equation (2), equation (12) is easily seen to be equivalent to the Schrödinger equation associated with the classical Lagrangian (6):

$$i\hbar(\partial_t-\tfrac{1}{4}\partial_t\ln|m|)\Psi=\tfrac{1}{2}m^{ij}(i\hbar\mathrm{D}_i+a_i)(i\hbar\mathrm{D}_j+a_j)\Psi+\mathcal{V}\Psi$$

for

$$\Psi=\rho^{1/2}|m|^{1/4}\exp(\mathrm{i}S/\hbar).$$

Let us restate the minor differences introduced in the general variational principle for arbitrary λ: in the conditions of variation, the two end points are kept fixed; although in the Markovian case, the function S can be obtained as a conditional expectation value of the stochastic action taken at initial time, this property is not really needed in the derivation, as a solution of equation (11) appears to be sufficient. Finally, one can remark that for any λ the stochastic action $\mathcal{S}_\lambda$ identifies with the usual Schrödinger action of quantum mechanics (up to terms depending on the endpoints only).

3. Classical and stochastic Lagrangians

We shall now recover a crucial feature of Guerra and Morato's scheme, i.e. that the Lagrangians introduced in (8) are not *ad hoc* assumptions, but can be computed from the classical Lagrangian (6). Before that, let us make a few comments to exhibit the remarkable character of this correspondence.

3.1. Correspondence principles

As in the case of the operator representation of quantum mechanics, the stochastic Lagrangians (8) for a particular system could be determined by a kind of correspondence principle, from the classical Lagrangian (6). Indeed, the stochastic action can be rewritten as

$$\mathcal{S}_\lambda=\left\langle\int_t^{t'}\left[\tfrac{1}{2}m_{ij}v^iv^j-\frac{\hbar^2}{8}m^{ij}\frac{\nabla_i\tilde{\rho}}{\tilde{\rho}}\frac{\nabla_j\tilde{\rho}}{\tilde{\rho}}+a_iv^i-\mathcal{V}\right]\mathrm{d}s\right\rangle$$

which could be obtained from the classical action by making the following substitutions:

$$\frac{\mathrm{d}x^i}{\mathrm{d}t}\to v^i \qquad \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\to\left(v^i+\frac{\hbar}{2}m^{ik}\frac{\nabla_k\tilde{\rho}}{\tilde{\rho}}\right)\left(v^j-\frac{\hbar}{2}m^{jl}\frac{\nabla_l\tilde{\rho}}{\tilde{\rho}}\right).$$

But, such a correspondence would still need to further justify the $\hbar$ corrections in the square of the velocity. More importantly, it would also remain 'global', as the velocity $v^i(x)$ at point x requires the knowledge of the whole process to be evaluated, so that the Lagrangian could not be defined for each trajectory in a universal way, i.e. with no dependence on the measure defining the process. These are precisely the new properties brought by the scheme, and which one would like to preserve. In the case of Markovian diffusions, Guerra and Morato (1983), (or Nelson 1985), have computed the mean value of classical Lagrangian (6), with $\mathrm{d}x^i/\mathrm{d}t$ taken to be the direct time derivative of the stochastic variable itself, or a covariant version of it taking into account the Itô terms:

$$\mathrm{d}x^i \to \tilde{\mathrm{d}}x^i_t = \mathrm{d}x^i_t + \tfrac{1}{2}\Gamma^i_{jk}\,\mathrm{d}x^j_t\,\mathrm{d}x^k_t + \tfrac{1}{6}(\nabla_l \Gamma^i_{jk} + \Gamma^i_{lm}\Gamma^m_{jk})\,\mathrm{d}x^j_t\,\mathrm{d}x^k_t\,\mathrm{d}x^l_t. \tag{13}$$

Of course, for diffusions the mean value of the classical action is generically infinite, but it can be shown in this case that the infinite part depends on the diffusion coefficient only and does not contribute to the variation. One can thus forget this infinite part (imagine a kind of renormalisation that will not depend on the drift field) and one is left with expression (8) where λ has been set equal to 1. The classical action thus defines a universal function, a length, on the sample paths or trajectories. This property (for Markovian diffusions) suggests that there should be a way to extend the classical Lagrangian into a unique stochastic one: by evaluating the former on all the sample paths and then taking the mean value over the process.

This uniqueness contrasts with the usual situation for path integrals, where ambiguities related to the different conventions of stochastic calculus (also related to different order prescriptions for operators) are known to affect the choice of a Lagrangian (Langouche *et al* 1979). But one can easily see that the stochastic action (7), which depends on the total mean value, will not be changed by different time prescriptions; or else, different substitutions, with $\mathrm{d}_\lambda x$ replacing $\mathrm{d}x$ in (6) and (13), will all lead, for Markovian diffusions, to the same result, which is the stochastic action obtained from (8) will λ set equal to 1 (or equivalently-1). (Anticipating the result of the next subsection, this already illustrates how the choice of a particular noise differs in nature from the choice of a convention in stochastic calculus.) Thus, one is led to think that only one stochastic Lagrangian can result from an intrinsic correspondence with classical dynamics. This in turn implies a unique Newton–Nelson law, and the uniqueness of the diffusion coefficient of the process one can associate with a quantum state.

But, as we shall demonstrate, this uniqueness is in fact the result of the implicit conditions on the noise correlations, which are imposed with the Markov hypothesis. Indeed, we shall evaluate the mean value of the classical action for general non-Markovian diffusions and obtain all the stochastic Lagrangians of (8), thus showing that the same intrinsic correspondence, using the classical action as a length for the sample paths, exists for general diffusions and in a transparent way with respect to the time-dependent parameter λ.

First we shall need a slightly improved classical action: the diffusive nature of the process (1) means that the trajectories have generic $\mathrm{d}x_t$ of order $\mathrm{d}t^{1/2}$, instead of $\mathrm{d}t$ in the classical limit. So that expressions with additional terms of the form

$$\frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k}{\mathrm{d}t} \qquad \frac{\mathrm{d}x^i\,\mathrm{d}x^j\,\mathrm{d}x^k\,\mathrm{d}x^l}{\mathrm{d}t} \qquad \mathrm{d}x^i\,\mathrm{d}x^j$$

known as Itô terms (MacLaughlin and Schulman 1971), will all have the same classical limit. One can see that such terms must in fact be present, in order to preserve the covariant scalar nature of the Lagrangian under (time-dependent) changes of coordinates, where $\mathrm{d}x_t$ is of order $\mathrm{d}t^{1/2}$. In fact, from the coefficients of the classical Lagrangian (m, a, $\mathscr{V}$) and their covariance properties, these Itô terms are determined up to a usual $R\,\mathrm{d}t$ term, where R is the curvature of the metrics m. One is led to the covariant length

(de Witt 1957)

$$\mathrm{d}s_t = \tfrac{1}{2} m_{ij} \frac{\tilde{\mathrm{d}} x_t^i \tilde{\mathrm{d}} x_t^j}{\mathrm{d}t} + a_i\, \tilde{\mathrm{d}} x_t^i + \tfrac{1}{4}(\partial_i m_{ij} + \mathrm{D}_i a_j + \mathrm{D}_j a_i)\, \tilde{\mathrm{d}} x_t^i\, \tilde{\mathrm{d}} x_t^j - \mathscr{V}\, \mathrm{d}t + \mathrm{o}(\mathrm{d}t). \tag{14}$$

For the sake of clarity, and because of the more complex expressions introduced by inhomogeneous diffusion coefficients, the evaluation of the mean length will be performed in two steps. First, it will be derived in the case of constant diffusion coefficient, to point up the role of the noise. Then, it will easily be generalised to arbitrary diffusion coefficients (or curved space), thus providing the required covariant description.

3.2. Evaluation of the mean length

As can be seen in (6), the computation of the mean value of the classical action will require a correct evaluation of the second-order correlations as in (1), but improved to second order in $\mathrm{d}t$. In the Markovian case, this is obtained by developing the stochastic differentials (3) up to order $\mathrm{d}t^{3/2}$. The same procedure will be used here, but with the stochastic differentials involving the λ noise (5), and also for arbitrary diffusions. Developing (5), one obtains

$$\mathrm{d}_\lambda x_t^i = \mathrm{d}_\lambda w_{\lambda t}^i + b_{\lambda t}^i\, \mathrm{d}t + \nabla_j b_{\lambda t}^i \int_{t_\lambda}^{t_\lambda + \mathrm{d}t} (w_{\lambda s}^j - w_{\lambda t}^j)\, \mathrm{d}s + \mathrm{o}(\mathrm{d}t^{3/2})$$

where the endpoints have been chosen so as to prepare for the expectation value, that will be taken conditionally in x_t^i; and straightforwardly:

$$\langle \mathrm{d}_\lambda x_t^i\, \mathrm{d}_\lambda x_t^j \rangle_t = \langle \mathrm{d}_\lambda w_{\lambda t}^i\, \mathrm{d}_\lambda w_{\lambda t}^j \rangle_t + b_{\lambda t}^i b_{\lambda t}^j\, \mathrm{d}t^2 + \nabla_k b_{\lambda t}^i \left\langle \mathrm{d}_\lambda w_{\lambda t}^j \int_{t_\lambda}^{t_\lambda + \mathrm{d}t} (w_{\lambda s}^k - w_{\lambda t}^k)\, \mathrm{d}s \right\rangle_t$$

$$+ \nabla_k b_{\lambda t}^j \left\langle \mathrm{d}_\lambda w_{\lambda t}^i \int_{t_\lambda}^{t_\lambda + \mathrm{d}t} (w_{\lambda s}^k - w_{\lambda t}^k)\, \mathrm{d}s \right\rangle_t + \mathrm{o}(\mathrm{d}t^2).$$

Then, the key, and only further hypothesis, will be that the correlations of the diffusion $\langle x_{t'}^i x_{t''}^j \rangle_t$ for different t' and t'', are assumed to be twice differentiable, i.e.

$$\partial_{t'} \partial_{t''} \langle x_{t'}^i x_{t''}^j \rangle_t \qquad \partial_{t'} \langle x_{t'}^i b_\lambda^j (x_{t''}) \rangle_t$$

exist and are continuous for any t, t', t'' all different, so that

$$\langle \mathrm{d} w_{\lambda t'}^i\, \mathrm{d} w_{\lambda t''}^j \rangle_t = c_{\lambda t}^{ij}(t', t'')\, \mathrm{d}t'\, \mathrm{d}t'' + \mathrm{o}(\mathrm{d}t'\, \mathrm{d}t'') \qquad t' \neq t'' \tag{15}$$

with

$$\begin{aligned} c_{\lambda t}^{ij}(t', t'') &= \partial_{t'} \partial_{t''} \langle w_{\lambda t'}^i w_{\lambda t''}^j \rangle_t, \qquad t' \neq t'' \\ &= \partial_{t'} \partial_{t''} \langle x_{t'}^i x_{t''}^j \rangle_t - \partial_{t'} \langle x_{t'}^i b_{\lambda t''}^j \rangle_t - \partial_{t''} \langle b_{\lambda t'}^i x_{t''}^j \rangle_t + \langle b_{\lambda t'}^i b_{\lambda t''}^j \rangle_t . \end{aligned}$$

Then, one easily computes

$$\langle \mathrm{d}_\lambda w_{\lambda t}^i\, \mathrm{d}_\lambda w_{\lambda t}^j \rangle_t = 2\nu_t^{ij}\, \mathrm{d}t + \left(\lambda \partial_t \nu_t^{ij} + \frac{1+\lambda^2}{2} \nu_t^{kl} \nabla_k \nabla_l \nu_t^{ij} + \lambda \nabla_k \nu_t^{ij} b_{\lambda t}^k + c_{\lambda t}^{ij} \right) \mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2) \tag{16}$$

where

$$\int_{t_\lambda}^{t_\lambda + \mathrm{d}t} \mathrm{d}t' \int_{t_\lambda}^{t_\lambda + \mathrm{d}t} \mathrm{d}t''\, c_{\lambda t}^{ij}(t', t'') = c_{\lambda t}^{ij}\, \mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2)$$

and also

$$\left\langle \mathrm{d}_\lambda w^i_{\lambda t} \int_{t_\lambda}^{t_\lambda+\mathrm{d}t} (w^j_{\lambda s} - w^j_{\lambda t})\,\mathrm{d}s \right\rangle_t = \lambda \nu^{ij}_t\,\mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2)$$

so that finally

$$\langle \mathrm{d}_\lambda x^i_t\,\mathrm{d}_\lambda x^j_t\rangle_t = 2\nu^{ij}_t\,\mathrm{d}t + \Big(b^i_{\lambda t} b^j_{\lambda t} + \lambda(\partial_t \nu^{ij}_t + \nu^{ik}_t \nabla_k b^j_{\lambda t} + \nu^{jk}_t \nabla_k b^i_{\lambda t} + \nabla_k \nu^{ij}_t b^k_{\lambda t})$$
$$+\frac{1+\lambda^2}{2}\,\nu^{kl}_t \nabla_k \nabla_l \nu^{ij}_t + c^{ij}_{\lambda t}\Big)\,\mathrm{d}t^2 + \mathrm{o}(\mathrm{d}t^2).$$

In the case of constant diffusion field, the length (14) reduces to

$$\mathrm{d}s_t = \tfrac{1}{2} m_{ij} \frac{\mathrm{d}x^i_t\,\mathrm{d}x^j_t}{\mathrm{d}t} + a_i\,\mathrm{d}x^i_t + \tfrac{1}{4}(\partial_t m_{ij} + \nabla_i a_j + \nabla_j a_i)\,\mathrm{d}x^i_t\,\mathrm{d}x^j_t - \mathscr{V}\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t)$$

and takes the following mean value (where identification (9) has been made):

$$\langle \mathrm{d}s_{t_\lambda}\rangle_t = \mathscr{L}_\lambda\,\mathrm{d}t + \frac{\hbar N}{2\lambda}\left(1 - \frac{\partial_t \lambda}{2}\,\mathrm{d}t\right) + \tfrac{1}{2} m_{ij} c^{ij}_{\lambda t}\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t)$$
$$\mathscr{L}_\lambda = \tfrac{1}{2} m_{ij} b^i_\lambda b^j_\lambda + a_i b^i_\lambda + \tfrac{1}{2}\hbar \nabla_i (b^i_\lambda + a^i) - \mathscr{V}$$

where $\mathscr{L}_\lambda$ is the stochastic Lagrangian (8).

Before examining the general case, a few remarks can be made. As previously noticed, the particular time prescription related to λ enters at two stages: in the convention for differentials (like d_λ), and in the definition of the noise w_λ. But it must be stressed that, although both have been used here for computational convenience, the first occurence (i.e. in d_λ) is inessential: the result does not depend on it, and if one uses d instead of d_λ one still obtains $\mathscr{L}_\lambda$, as is obvious from the following identiy:

$$\mathscr{S} = \left\langle \int_t^{t'} \mathrm{d}s_{t_\lambda} \right\rangle = \left\langle \int_t^{t'} \mathrm{d}s_t \right\rangle.$$

Only the second occurence of λ plays a role, and results in different breakings of $\mathscr{S}$ into the sum of two parts: one giving the stochastic Lagrangian, and the other being a noise contribution. To be more explicit, one can also compute the difference in the noise correlations for various λ and the same process. Using the differentiability (15) and

$$\mathrm{d}w^i_{\mu t} = \mathrm{d}w^i_{\lambda t} + (\lambda - \mu)\frac{\nabla_j(\nu^{ij}\rho)}{\rho}\,\mathrm{d}t + \mathrm{o}(\mathrm{d}t)$$

one obtains quite generally

$$\langle c^{ij}_{\mu t}\rangle = \langle c^{ij}_{\lambda t}\rangle + (\mu^2 - \lambda^2)\left\langle \frac{\nabla_k(\nu^{ik}\rho)}{\rho}\,\frac{\nabla_l(\nu^{jl}\rho)}{\rho} \right\rangle$$

which is to be compared with (8), or

$$\langle \mathscr{L}_\lambda \rangle = \left\langle \tfrac{1}{2} m_{ij}\left(v^i v^j - \lambda^2 \frac{\nabla_k(\nu^{ik}\rho)}{\rho}\,\frac{\nabla_l(\nu^{jl}\rho)}{\rho}\right) + a_i v^i - \mathscr{V} \right\rangle.$$

This explicitly shows how different stochastic Lagrangians are generated. In particular, this explains why the Markovian case, $c^{ij}_{1t} = 0$, leads to a unique stochastic action (the one with $\lambda = 1$). The equivalent variational principles of the previous section can now be better understood. They correspond to extremising the same universal function, the mean length of the trajectories, but for different diffusion coefficients and with different conditions of variation, i.e. while keeping fixed different (infinite) parts related to different noises. They thus provide different critical diffusions, which are all related to the same Schrödinger equation. Moreover, this shows in a clear way that the correlations of the noise need not be specified: the functions $c^{ij}_{\lambda t}$ in (15) can be arbitrary. They may just be kept fixed during variation. In particular, this is precisely the effect of the Markovian property, which amounts to specifying $c^{ij}_{\lambda t}=0$, and using $\lambda=1$. In fact, only the integral:

$$\mathscr{C}_\lambda=\left\langle\int_t^{t'}\tfrac{1}{2}m_{ij}c^{ij}_{\lambda s}\,\mathrm{d}s\right\rangle \tag{17}$$

has to be fixed. By varying λ while keeping (17) constant, for the same drift field and probability density, one will generate a very large class of diffusions that all correspond to the same quantum state, with its Schrödinger evolution. In particular, among diffusions with $c^{ij}_{\lambda t}=0$, besides those which will be Markovian for $\lambda=1$, those for $\lambda\neq 1$ will constitute other equally acceptable diffusions (concerning the singular $\lambda=0$ case, let us just remark that it requires an infinite diffusion coefficient to be dealt with correctly, and thus does not correspond to a classical limit). One might wonder whether such diffusions actually exist, and really build a class within which the variational principle can be applied. Indeed, the case of Gaussian processes can easily be worked out, and the various equivalent diffusions explicitly built, which are solutions of the previous stochastic variational principles. It thus appears that the degeneracy in the correspondence between quantum states and stochastic processes can be ultimately related to the properties of the noise which characterise the process.

3.3. *Time-dependent covariance*

The previous derivation will hold for any diffusion field (and not only for constant ones), if only because of general covariance. This will be naturally exhibited if one manages to use covariant objects only.

As remarked by Itô (1975), Dohrn and Guerra (1978) and Nelson (1985), diffusions bear a natural Riemannian structure, with a metric given by the diffusion coefficient ν. This can, moreover, be extended to include covariance under general time-dependent changes of variables (Jaekel 1988). In particular, the following expression generalises $\mathrm{d}x^i_t$ to a vector up to order $\mathrm{d}t^{3/2}$:

$$\mathrm{D}_a x^i_t=\tilde{\mathrm{d}}x^i_t+a^i\,\mathrm{d}t+\mathrm{D}_j a^i\tilde{\mathrm{d}}x^j_t\,\mathrm{d}t \tag{18}$$

$-a$ is a velocity, i.e. under time-dependent changes of coordinates:

$$t=f(\bar{t})\qquad \dot{f}=\partial_{\bar{t}}f\qquad x^i=g^i(\bar{t},\bar{x})$$

$$g'^i_j=\bar{\nabla}_j g^i\qquad \dot{g}^i=\partial_{\bar{t}}g^i(\bar{t},\bar{x})$$

$$-a^i=\frac{1}{\dot{f}}(-g'^i_j\bar{a}^j+\dot{g}^i)$$

$$\mathrm{D}_a x^i_t=g'^i_j\mathrm{D}_{\bar{a}}\bar{x}^j_{\bar{t}}+\mathrm{o}(\mathrm{d}t^{3/2}).$$

$\tilde{\mathrm{d}}x_t$, (13), is in fact the expression of the tangent vector at x_t to the geodesic running from x_t to $x_t+\mathrm{d}x_t$. Similarly, the metric is generalised to the following tensor up to order $\mathrm{d}t$:

$$m^a_{ij}=\frac{m_{ij}}{\mathrm{d}t}+\tfrac{1}{2}(\partial_t m_{ij}-\mathrm{D}_i a_j-\mathrm{D}_j a_i)$$

leading to the scalar length

$$\mathrm{d}s_t=\tfrac{1}{2}m^a_{ij}\mathrm{D}_a x^i_t \mathrm{D}_a x^j_t \tag{19}$$

which gives expression (14).

As shown in the previous section, one will also need, for evaluating the expectation value of the correlations of the noise, to know the latter up to order $\mathrm{d}t^{3/2}$ only. It will be convenient to introduce a generalised λ-noise $\tilde{w}_\lambda$ by

$$\mathrm{d}\tilde{w}^i_{\lambda t}=\mathrm{d}x^i_t-\int_t^{t+\mathrm{d}t}\tilde{b}^i_{\lambda s}\,\mathrm{d}s=dw^i_{\lambda t}-\lambda\nu^{jk}\Gamma^i_{jk}\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t) \tag{20}$$

and the two corresponding vectors up to order $\mathrm{d}t$:

$$\tilde{\mathrm{d}}\tilde{w}^i_{\lambda t}=\mathrm{d}\tilde{w}^i_{\lambda t}+\tfrac{1}{2}\Gamma^i_{jk}\,\mathrm{d}\tilde{w}^j_{\lambda t}\,\mathrm{d}\tilde{w}^k_{\lambda t}$$

$$\tilde{\mathrm{d}}_*\tilde{w}^i_{\lambda t}=\mathrm{d}_*\tilde{w}^i_{\lambda t}-\tfrac{1}{2}\Gamma^i_{jk}\,\mathrm{d}_*\tilde{w}^j_{\lambda t}\,\mathrm{d}_*\tilde{w}^k_{\lambda t}$$

with the following property of the correlations (resulting from (15)):

$$\langle\tilde{\mathrm{d}}\tilde{w}^i_{\lambda t'}\,\tilde{\mathrm{d}}_*\tilde{w}^j_{\lambda t''}\rangle_t=\tilde{c}^{ij}_{\lambda t}(t',t'')\,\mathrm{d}t'\,\mathrm{d}t''+\mathrm{o}(\mathrm{d}t'\,\mathrm{d}t'')\qquad t'>t''. \tag{21}$$

Equation (21) then implies that $\tilde{c}^{ij}_{\lambda t}$ is a tensor:

$$\int_{t_\lambda}^{t_\lambda+\mathrm{d}t}\mathrm{d}t'\int_{t_\lambda}^{t_\lambda+\mathrm{d}t}\mathrm{d}t''\,\tilde{c}^{ij}_{\lambda t}(t',t'')=\tilde{c}^{ij}_{\lambda t}\,\mathrm{d}t^2+\mathrm{o}(\mathrm{d}t^2). \tag{22}$$

Having identified the different tensors, one can choose normal coordinates (such that $\Gamma^i_{jk}=0$ at x_t) and compute the required correlations using the same development as in the previous section. One then obtains

$$\langle\nu_{ij t_\lambda}\,\tilde{\mathrm{d}}x^i_{t_\lambda}\,\tilde{\mathrm{d}}x^j_{t_\lambda}\rangle_t=2N\,\mathrm{d}_\lambda t+(\nu_{ij}\tilde{b}^i_\lambda\tilde{b}^j_\lambda+2\lambda\,\mathrm{D}_i\tilde{b}^i_\lambda+\partial_t\ln|\nu|-\tfrac{2}{3}R+\nu_{ij}\tilde{c}^{ij}_{\lambda t})\,\mathrm{d}_\lambda t^2+\mathrm{o}(\mathrm{d}_\lambda t^2).$$

Here, the subscript λ attached to the differential $\mathrm{d}_\lambda t$ refers to the transformation properties of the latter, i.e. one must define $\mathrm{d}_\lambda t$ more precisely as

$$\mathrm{d}t_\lambda=\left(t+\frac{1+\lambda}{2}\,\mathrm{d}_\lambda t\right)-\left(t-\frac{1-\lambda}{2}\,\mathrm{d}_\lambda t\right)$$

up to order $\mathrm{d}t^2$, such that it transforms as

$$\mathrm{d}_\lambda t=\dot{f}\,\mathrm{d}_\lambda\bar{t}+\frac{\lambda}{2}\ddot{f}\,\mathrm{d}_\lambda\bar{t}^2+\mathrm{o}(\mathrm{d}_\lambda\bar{t}^2).$$

Identifying ν with $\hbar m/2\lambda$, we obtain

$$\langle\mathrm{d}s_{t_\lambda}\rangle_t=\frac{\hbar N}{2\lambda}\left(1-\frac{\partial_t\lambda}{2}\,\mathrm{d}t\right)+\mathscr{L}_\lambda\,\mathrm{d}t-\frac{\hbar}{6\lambda}R\,\mathrm{d}t+\tfrac{1}{2}m_{ij}\tilde{c}^{ij}_{\lambda t}\,\mathrm{d}t+\mathrm{o}(\mathrm{d}t)$$

where $\mathscr{L}_\lambda$ is again the stochastic Lagrangian used in section 2, and R is the scalar curvature of the metrics:

$$R=m^{kl}(\nabla_i\Gamma^i_{kl}-\nabla_l\Gamma^i_{ik}+\Gamma^i_{im}\Gamma^m_{kl}-\Gamma^i_{ml}\Gamma^m_{ik}).$$

This also leads to the covariant action ($\tilde{\mathscr{C}}_\lambda = \int_t^{t'} \frac{1}{2} m_{ij} \tilde{c}^{ij}_{\lambda s}\, \mathrm{d}s$):

$$\mathscr{S} = \int_t^{t'} \frac{\hbar N}{2\lambda}\left(1 - \frac{\partial_t \lambda}{2}\,\mathrm{d}t\right) + \left\langle \int_t^{t'} \left(\mathscr{L}_\lambda - \frac{\hbar}{6\lambda} R\right) \mathrm{d}t \right\rangle + \tilde{\mathscr{C}}_\lambda .$$

The discussion of the previous section holds here: extremising the same mean length of the trajectories (14), on different diffusions (characterised by different diffusion coefficients $\hbar m/2\lambda$), but with appropriate conditions of variations, i.e. by keeping fixed the diffusion coefficient and the correlation (22) of the corresponding λ-noise, leads in all cases to the same and unique Schrödinger equation. A small difference is now that the λ-noise itself does indeed vary (as can be seen in its correlations up to order $\mathrm{d}t^2$(16)), but the expression $\tilde{\mathscr{C}}_\lambda$, which plays the role of a zero-point action, must be kept fixed. In particular, the Markov property corresponds to further constraining $\tilde{\mathscr{C}}_\lambda$ to be kept equal to zero.

4. Conclusion and outlook

One result of the scheme developed here is that the correspondence between quantum states and stochastic processes suffers from two kinds of degeneracies: firstly, the diffusion coefficient can be scaled by an arbitrary time-dependent parameter, provided the latter also modifies the value of the time taken for defining a decomposition between drift and noise (5). More precisely, the same expressions for the correlations must be attributed to a noise whose definition varies with the parameter λ. Thus, a change in the scale of the diffusion can be compensated for by a shift of the time involved in conditional expectation values (and hence by a change in the definition of the noise). Secondly, the noise need not be specified completely in order to determine the quantum state. While the singular part of its correlations is fixed by the metric of the Lagrangian, the regular part (21) can remain arbitrary without affecting the quantum evolution. One can regard the quantum state as representing the part of the information lying in the stochastic process which is relevant for the evolution of the system. The remaining part concerns the noise and is, to some extent, transparent to the Schrödinger equation. In fact, this property could be used to characterise pure states: the evolution of mixed states is easily recovered by mixing the corresponding diffusion processes (Jaekel and Pignon 1984); using the same argument as therein, one can easily see that the correct variational principle will be obtained not by keeping the total contribution of the λ-noise fixed, but by keeping it fixed for each pure component.

On the whole, the scheme still supports the view of the stochastic representation that classical mechanics can be extended to quantum mechanics by maintaining the notion of trajectories, and replacing their differentiability ($\mathrm{d}x \sim \mathrm{d}t$) by a diffusive behaviour ($\mathrm{d}x \sim \mathrm{d}t^{1/2}$). In doing so, as in the path integral formalism, the stochastic representation of quantum mechanics introduces a new kind of relationship between kinematics and dynamics: the diffusion field, which characterises the size of the fluctuations of the sample paths, appears to be linked to the metrics defining the kinetic part of the Lagrangian (albeit up to an overall scaling time-dependent parameter). But the stochastic formalism does not throw much light on this relation. A possibility for a dynamical mechanism might be in the curvature term that enters the mean length of the trajectories. Associating it with the action of the metrics could provide a basis for a complementary variational principle involving the diffusion field and the metrics (Guerra and Morato 1983). The stochastic variational principle studied here only relates the time-dependent scaling arbitrariness to a choice of time and a related choice of a decomposition between drift and noise. This property might make the stochastic representation particularly pertinent to problems raised by the notion of time in quantum mechanics. For instance, circumstances related to multiple-time observables have already been discussed, showing that the path formalism provides a broader description than that of states (Aharonov and Albert 1984). Also, if one looks for an operational definition of time, such as that given by quantum clocks, there appear difficulties to understanding what is meant by differentials with respect to time in the Schrödinger equation within the standard operator representation (Peres 1980). The description in terms of equivalent diffusions, while preserving its geometric roots to the notion of trajectory, might also show sufficient flexibility to agree with the quantum requirements.

The other result that arbitrary noise correlations still lead to the Schrödinger evolution, when extended to include mixed states, could provide the basis for a stochastic characterisation of pure states. This should be compared with the description given by the quantum Langevin equation (Ford *et al* 1965), where the general quantum regime also corresponds to noise correlations which are not Markovian, and become so only in the high temperature limit. The associated quantum process contains, besides information on the small system's evolution, information on the bath which is described by the noise correlations. Moreover, a purely stochastic representation of the quantum Langevin equation provides a common framework for treating 'quantum' and 'statistical' fluctuations on an equal footing (Jaekel 1989). A unified approach might prove fruitful for understanding the peculiar character of quantum evolution, as opposed to ordinary diffusions.

References

Aharonov Y and Albert D Z 1984 *Phys. Rev.* D **29** 223
Davidson M 1979 *Physica* **96A** 465
de Witt B S 1957 *Rev. Mod. Phys.* **29** 377
Dohrn D and Guerra F 1978 *Lett. Nuovo Cimento* **22** 121
Ford G W, Kac M and Masur P 1965 *J. Math. Phys.* **6** 504
Guerra F and Morato L M 1983 *Phys. Rev.* D **27** 1774
Itô K 1975 *Probabilistic Methods in Differential Equations* (*Lecture Notes in Math.* **451**) ed M A Pinsky (Berlin: Springer) p 1
Jaekel M T 1988 *J. Physique* **49** 927
—— 1989 *J. Phys. A: Math. Gen.* **22** 537
Jaekel M T and Pignon D 1984 *J. Phys. A: Math. Gen.* **17** 131
Langouche F, Roekaerts D and Tirapegui E 1979 *Phys. Rev.* D **20** 419
MacLaughlin D W and Schulman L S 1971 *J. Math. Phys.* **12** 2520
Marra R 1987 *Phys. Rev.* D **36** 1724
Misawa T and Yasue K 1987 *J. Math. Phys.* **28** 2569
Morato L M 1985 *Phys. Rev.* D **31** 1982
Nelson E 1966 *Phys. Rev.* **150** 1079
—— 1985 *Quantum Fluctuations* (Princeton, NJ: Princeton University Press)
Peres A 1980 *Am. J. Phys.* **48** 552
Schuss Z 1980 *Theory and Applications of Stochastic Differential Equations* (New York: Wiley)
Yasue K 1981 *J. Funct. Anal.* **41** 327